Tutorial 9

Haowei Qi

March 27, 2025

Contents

1 Question 1: §8.1 Q7

Evaluate $\lim(e^{-nx})$ for $x \in \mathbb{R}$, $x \geq 0$.

Proof. It converges pointwise to f on $x \geq 0$ where

$$f = \begin{cases} 1 & x = 0 \\ \\ 0 & x > 0 \end{cases}$$

But it is divergent on $x < 0$.

For any $\epsilon > 0$ and $x > 0$, let $N = [\frac{\ln \frac{1}{\epsilon}}{x}] + 1$.Then $|e^{-nx} - 0| \leq e^{-\ln(\frac{1}{\epsilon})} = \epsilon$ for any $n > N$. For $x = 0$, $e^{-n \cdot 0} = 1 = f(0)$. But it doesn't converge uniformly.

By Lemma 8.1.5, Let $\epsilon = \frac{1}{2e}$, $x_n = \frac{1}{n}$. Then $f_n(x_n) = \frac{1}{e} > \frac{1}{2e}$ for all n.

By Lemma 8.1.8, $\|e^{-nx} - f\|_{\{x \geq 0\}} = \sup\{|e^{-nx} - f| : x \geq 0\} = 1$

For any given $x < 0$, $\lim_{n\to\infty} e^{-nx} = \infty$.

□

2 Question 2: §8.1 Q17

Show that if $a > 0$, then the convergence of the sequence in Exercise 7 is uniform in the interval $[a, \infty)$, but not uniform in the interval $[0, \infty)$.

Proof. We have already proved that it is not uniform on interval $[0, \infty)$.

By definition, fix $a > 0$. By definition, for any $\epsilon > 0$, let $N = [\frac{\ln \frac{1}{\epsilon}}{a}] + 1$, then

$$|e^{-nx} - 0| \leq \epsilon$$

for $n > N$ and $x \geq a$.

By Lemma 8.1.8, $\|e^{-nx}\|_{\{x \geq a\}} = e^{-na}$. Hence $\lim \|e^{-nx}\|_{\{x \geq a\}} = 0$ and it converges uniformly. □

3 Question 3: §8.1 Q21

Show that if (f_n), (g_n) converge uniformly on the set A to f, g, respectively, then $(f_n + g_n)$ converges uniformly on A to $f + g$.

Proof. By definition, for any $\epsilon > 0$, there exist N_1 and N_2, such that $|f_n - f| < \epsilon/2$ for $n > N_1$ and $x \in A$, as well as $|g_n - g| < \epsilon/2$ for $n > N_2$ and $x \in A$. Let $N = \max N_1, N_2$. Then $|f_n+g_n-(f+g)| \leq |f_n-f|+|g_n-g| < \epsilon$ for $n > N$ and $x \in A$. Hence $f_n+g_n \to f+g$ uniformly.

By Lemma 8.1.8, since $\|f_n+g_n-(f+g)\| \leq \|f_n-f\|+\|g_n-g\|$, $\lim \|f_n+g_n-(f+g)\| \leq \lim \|f_n - f\| + \lim \|g_n - g\| = 0$. Hence $f_n + g_n \to f + g$ uniformly. □

4 Question 4: §8.2 Q7

Suppose f_n converges to f on the set A, and suppose the each f_n is bounded on A. (That is, each n there is a constant M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$.) show that the function f is bounded on A.

Proof. Since f_n converges,s uniformly on A, for any $\epsilon > 0$, there exists an N such that $|f_n - f| \leq \epsilon$ for all $x \in A$ and $n \geq N$.

Thus,

$$|f| \leq |f_N| + |f_N - f| \leq M_N + \epsilon,$$

for all $x \in A$.

Hence, f is a bounded function. □

5 Question 5: §8.2 Q13

If $a > 0$, show that $\lim \int_a^\pi \frac{\sin nx}{nx}\,\mathrm{d}x = 0$. What if $a = 0$.

Proof. Since $a > 0$,

$$\|\frac{\sin nx}{nx}\|_{[a,\pi]} \leq \|\frac{1}{nx}\|_{[a,\pi]} = \frac{1}{an} \to 0.$$

Hence $\frac{\sin nx}{nx}$ converges to 0 uniformly. Since $\frac{\sin nx}{nx}$ is continuous on $[a,\pi]$, $\frac{\sin nx}{nx} \in \mathcal{R}[a,\pi]$. Hence, by Theorem 8.2.4, $0 = \int_a^\pi 0\,\mathrm{d}x = \lim \int_a^\pi \frac{\sin nx}{nx}\,\mathrm{d}x$.

Since $\frac{\sin nx}{nx}$ is continuous on $(0,\pi]$, it is sufficient to check whether it is right-continuous at 0. By L'Hôpital's rule, $\frac{\sin nx}{nx}|_{x=0} = 1$ same as $\lim_{x\to x^+} \frac{\sin nx}{nx}$. Thus, $\frac{\sin nx}{nx} \in \mathcal{R}[0,\pi]$.

Since $|\frac{\sin nx}{nx}| \leq 1$, by Bounded Convergence Theorem, $0 = \lim \int_0^\pi \frac{\sin nx}{nx}\,\mathrm{d}x$. □